Appendix B: Stochastic model solution

In this appendix we solve the stochastic version of the model. To do this, we start by linearizing the model around its steady state. Despite the simplicity of the structure of the proposed DSGE model, it is highly nonlinear, reflecting very complex relationships between different economic variables. This hampers their practical application. To solve this problem, we resort to performing a linear approximation to the equations of the model, which would allow us to direct application to the data.

The log-linearization of the model consists in expressing the variables as log-linear deviations with respect to their steady state values. The log-linear deviation of a variable u around its steady state, $\overline{u}$, is denoted as $\widehat{u}$, where $\widehat{u}_t = \ln u_t - \ln \overline{u}$. That is

$$u_t = \overline{u}e^{\widehat{u}_t} \approx \overline{u}(1 + \widehat{u}_t)$$

In constructing the log-linear deviations we follow two basic rules (Uhlig, 1999). First, for the case of two variables u_t and z_t, we have:

$$u_t z_t \approx \overline{u}(1 + \widehat{u}_t)\overline{z}(1 + \widehat{z}_t) \approx \overline{uz}(1 + \widehat{u}_t + \widehat{z}_t)$$

that is, we assume that the product of the two deviations, i.e., $\widehat{u}_t\widehat{z}_t$, is approximately equal to zero, as they are small numbers. Second, we assume the following approximation:

$$u_t^a \approx \overline{u}^a(1 + \widehat{u}_t)^a \approx \overline{u}^a(1 + a\widehat{u}_t)$$

Taking into account the above definitions, we can proceed to the log-linearization of our model. We start from the production function:

$$Y_t = A_t K_t^\alpha L_t^{1-\alpha}$$

In steady state, the production function can be written as:

$$\overline{Y} = \overline{A}\overline{K}^\alpha \overline{L}^{1-\alpha}$$

Therefore, using the above basic rules, we can write:

$$\overline{Y}(1 + \widehat{y}_t) = \overline{A}\overline{K}^\alpha \overline{L}^{1-\alpha}(1 + \widehat{a}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{l}_t)$$

Substituting, we obtain the log-linear equation for the production

function:

$$\widehat{y}_t = \widehat{a}_t + \alpha \widehat{k}_t + (1-\alpha)\widehat{l}_t \tag{A.1}$$

This procedure must be applied to the other equations of the model. For instance, the second equation we consider is:

$$C_t = Y_t - I_t$$

By calculating the deviation with respect to the steady state we obtain:

$$\widehat{c}_t = \frac{\overline{Y}}{\overline{C}}\widehat{y}_t - \frac{\overline{I}}{\overline{C}}\widehat{i}_t$$

Substituting the steady state values in the feasibility condition of the economy, we obtain:

$$\left[1 - \beta + (1-\alpha)\beta\delta\right]\widehat{c}_t = (1 - \beta + \beta\delta)\widehat{y}_t - \alpha\beta\delta\widehat{i}_t \tag{A.2}$$

The log-linear version of the capital stock accumulation equation is given by:

$$\widehat{k}_{t+1} = (1-\delta)\widehat{k}_t + \delta\widehat{i}_t \tag{A.3}$$

Next equation of the model is the following:

$$\frac{1-\gamma}{\gamma}\frac{C_t}{1-L_t} = (1-\alpha)\frac{Y_t}{L_t}$$

and after the necessary transformation we obtain:

$$(1-\gamma)\overline{CL} = \gamma(1-\alpha)\overline{Y}(1-\overline{L})$$

Again, substituting the steady state values previously computed, we obtain the following expression:

$$\left[1 + \frac{\gamma(1-\alpha)}{(1-\gamma)}\frac{1-\beta+\beta\delta}{1-\beta+(1-\alpha)\beta\delta}\right]\widehat{l}_t = \widehat{y}_t - \widehat{c}_t \tag{A.4}$$

The next equation is:

$$\frac{E_t C_{t+1}}{C_t} = E_t\beta\left[1 + \left(\alpha\frac{Y_{t+1}}{K_{t+1}} - \delta\right)\right]$$

and applying the same procedure, we obtain the following expression:

$$E_t \widehat{c}_{t+1} - \widehat{c}_t = \left(1 - \beta + \beta\delta\right) E_t \widehat{y}_{t+1} - \left(1 - \beta + \beta\delta\right) E_t \widehat{k}_{t+1} \qquad \text{(A.5)}$$

Finally, given our assumption that the TFP follows an AR(1) process, the log-deviation with respect to the steady state is given by:

$$\widehat{a}_t = \rho \widehat{a}_{t-1} + \varepsilon_t \qquad \text{(A.6)}$$

Once we have the model in log-linear form, we can proceed with its resolution, although we have to bear in mind that this is an approximation of the original highly nonlinear model. The literature had proposed different alternative methods to solve a DSGE model. These methods are the proposed by Blanchard and Kahn (1980), Uhlig (1999), Sims (2001) and Klein (2000). Here, we use the procedure developed by Blanchard and Kahn (1980). We follow Ireland (2004) in applying Blanchard-Kahn method. We start by defining the following two vectors of deviations from the steady state:

$$x_t^0 = \left[\begin{array}{c} \widehat{y}_t \\ \widehat{i}_t \\ \widehat{l}_t \end{array} \right] \qquad \text{(A.7)}$$

$$s_t^0 = \left[\begin{array}{c} \widehat{k}_t \\ \widehat{c}_t \end{array} \right] \qquad \text{(A.8)}$$

where the first vector comprises deviations in production, investment, and employment from their steady state valuesand the second vector is formed by the deviations of the capital stock and consumption, the variables for which we have not only its current value but also future value.

First, we can write the following system:

$$A x_t^0 = B s_t^0 + C \widehat{a}_t \qquad \text{(A.9)}$$

consisting of the following three equations:

$$\widehat{y}_t - (1 - \alpha)\widehat{l}_t = \widehat{a}_t + \alpha \widehat{k}_t$$

$$(1 - \beta + \beta\delta)\widehat{y}_t - \alpha\beta\delta\widehat{i}_t = \left[1 - \beta + (1 - \alpha)\beta\delta\right]\widehat{c}_t$$

$$\widehat{y}_t - \left[1 + \frac{\gamma(1-\alpha)}{(1-\gamma)} \frac{1-\beta+\beta\delta}{1-\beta+(1-\alpha)\beta\delta}\right]\widehat{l}_t = \widehat{c}_t$$

To simplify notation, we define the following three parameters:

$$\theta = 1 - \beta + \beta\delta$$

$$\phi = 1 - \beta + (1-\alpha)\beta\delta$$

$$\eta = 1 + \frac{\gamma(1-\alpha)}{(1-\gamma)} \frac{\theta}{\phi}$$

and where the constant matrices are given by:

$$A = \begin{bmatrix} 1 & 0 & \alpha-1 \\ \theta & \phi-\theta & 0 \\ 1 & 0 & -\eta \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha & 0 \\ 0 & \phi \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We also define the following system in terms of the expected future value of the variables in the model:

$$DE_t s^0_{t+1} + FE_t x^0_{t+1} = Gs^0_t + Hx^0_t \tag{A.10}$$

consisting in the following two equations:

$$\left(1-\beta+\beta\delta\right)E_t\widehat{k}_{t+1} + E_t\widehat{c}_{t+1} - \left(1-\beta+\beta\delta\right)E_t\widehat{y}_{t+1} = \widehat{c}_t$$

$$\widehat{k}_{t+1} = (1-\delta)\widehat{k}_t + \delta\widehat{i}_t$$

where the matrices as given by:

$$D = \begin{bmatrix} \theta & 1 \\ 1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} -\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1-\delta \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta & 0 \end{bmatrix}$$

Finally, the matrix model is closed by incorporating the expected deviation of total factor productivity:

$$E_t \widehat{a}_{t+j} = \rho_A^j \widehat{a}_t$$

The system (A.9) can be written as:

$$x_t^0 = A^{-1} B s_t^0 + A^{-1} C \widehat{a}_t$$

Taking one period ahead, the above system should be:

$$E_t x_{t+1}^0 = A^{-1} B E_t s_{t+1}^0 + A^{-1} C \rho_A \widehat{a}_t$$

Substituting in the system (A.10) we find that:

$$(D + FA^{-1}B) E_t s_{t+1}^0 = (G + HA^{-1}B) s_t^0 + (HA^{-1}C - FA^{-1}C\rho_A) \widehat{a}_t$$

Solving for the matrices, the final system would be:

$$E_t s_{t+1}^0 = J s_t^0 + M \widehat{a}_t$$

where:

$$J = (D + FA^{-1}B)^{-1} (G + HA^{-1}B)$$

$$M = (D + FA^{-1}B)^{-1} (HA^{-1}C - FA^{-1}C\rho_A)$$

Using the Jordan decomposition, the matrix J can be decomposed such as:

$$J = O^{-1} N O$$

where:

$$N = \begin{bmatrix} N_{11} & 0 \\ 0 & N_{22} \end{bmatrix}$$

and where:

$$O = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}$$

Notice that the elements of the diagonal of N are the eigenvalues of the matrix J. In order the solution to be unique, the value of N_{11} must be inside the unit circle and the value of N_{22} outside the unit circle. This is the so-called the Blanchard-Kahn rank condition. If the rank condition does not hold, then the equilibrium is not unique. The columns of O^{-1} are the eigenvectors of the matrix J. Therefore, the system can be written as:

$$E_t s_{t+1}^0 = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & 0 \\ 0 & N_{22} \end{bmatrix} \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} s_t^0 + \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \widehat{a}_t$$

$$\begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} E_t s_{t+1}^0 = \begin{bmatrix} N_{11} & 0 \\ 0 & N_{22} \end{bmatrix} \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} s_t^0$$
$$+ \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \widehat{a}_t$$

Alternatively, we write the following expectations:

$$E_t s_{1,t+1}^1 = N_{11} s_{1,t}^1 + Q_{11} \widehat{a}_t$$

$$E_t s_{2,t+1}^1 = N_{22} s_{2,t}^1 + Q_{21} \widehat{a}_t$$

where:

$$s_{1,t}^1 = O_{11} \widehat{k}_t + O_{12} \widehat{c}_t$$

$$s_{2,t}^1 = O_{21} \widehat{k}_t + O_{22} \widehat{c}_t$$

and where:

$$Q = \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} = OM$$

Given that the value of N_{22} is outside the unit circle, we can solve $s^1_{2,t}$ ahead:

$$s^1_{2,t} = \frac{1}{N_{22}} E_t s^1_{2,t+1} - \frac{Q_{21}}{N_{22}} \widehat{a}_t$$

resulting:

$$
\begin{aligned}
s^1_{2,t} &= -\frac{Q_{21}}{N_{22}} \sum_{j=0}^{\infty} \left(\frac{1}{N_{22}} \right)^j E_t \widehat{a}_{t+j} \\
&= -\frac{Q_{21}}{N_{22}} \sum_{j=0}^{\infty} \left(\frac{\rho_A}{N_{22}} \right)^j \widehat{a}_t = \frac{Q_{21}}{\rho_A - N_{22}} \widehat{a}_t
\end{aligned}
$$

Solving for $\widehat{c}_t$ we obtain:

$$\frac{Q_{21}}{\rho_A - N_{22}} \widehat{a}_t = O_{21} \widehat{k}_t + O_{22} \widehat{c}_t$$

Thus, the log-deviation of consumption is:

$$\widehat{c}_t = -\frac{O_{21}}{O_{22}} \widehat{k}_t + \frac{Q_{21}}{O_{22}(\rho_A - N_{22})} \widehat{a}_t$$

or alternatively:

$$\widehat{c}_t = S_1 \widehat{k}_t + S_2 \widehat{a}_t$$

being

$$S_1 = -\frac{O_{21}}{O_{22}}$$

$$S_2 = \frac{Q_{21}}{O_{22}(\rho_A - N_{22})}$$

In the case of the vector $s^1_{1,t}$ we find that:

$$s^1_{1,t} = (O_{11} + O_{12} S_1) \widehat{k}_t + O_{12} S_2 \widehat{c}_t$$

and substituting we obtain:

$$E_t s^1_{1,t+1} = N_{11} s^1_{1,t} + Q_{11} \widehat{a}_t$$

$$E_t s^1_{1,t+1} = N_{11} \left[(O_{11} + O_{12} S_1) \widehat{k}_t + O_{12} S_2 \widehat{c}_t \right] + Q_{11} \widehat{a}_t$$

$$(O_{11} + O_{12} S_1) \widehat{k}_{t+1} = N_{11} (O_{11} + O_{12} S_1) \widehat{k}_t + (Q_{11} + O_{12} S_2 (1 - \rho_A)) \widehat{a}_t$$

or alternatively:

$$\widehat{k}_{t+1} = S_3 \widehat{k}_t + S_4 \widehat{a}_t$$

where:

$$S_3 = N_{11}$$

$$S_4 = \frac{Q_{11} + N_{11} O_{12} S_2 - O_{12} S_2 \rho_A}{O_{11} + O_{12} S_1}$$

Finally, returning to the initial system:

$$x^0_t = A^{-1} B s^0_t + A^{-1} C \widehat{a}_t$$

$$x^0_t = A^{-1} B \left[\begin{array}{c} \widehat{k}_t \\ \widehat{c}_t \end{array} \right] + A^{-1} C \widehat{a}_t$$

$$x^0_t = A^{-1} B \left[\begin{array}{c} 1 \\ S_1 \end{array} \right] \widehat{k}_t + \left[A^{-1} C + A^{-1} B \left[\begin{array}{c} 0 \\ S_2 \end{array} \right] \right] \widehat{a}_t$$

or:

$$x^0_t = S_5 s^0_t + S_6 \widehat{a}_t$$

where:

$$S_5 = A^{-1} B \left[\begin{array}{c} 1 \\ S_1 \end{array} \right]$$

$$S_6 = A^{-1} C + A^{-1} B \left[\begin{array}{c} 0 \\ S_2 \end{array} \right]$$

Having completed all these computations, the solution of the model can be obtained. Collecting terms, the solution of the model is

given by:

$$
\begin{bmatrix} \widehat{k}_{t+1} \\ \widehat{a}_{t+1} \end{bmatrix} = \begin{bmatrix} S_3 & S_4 \\ 0 & \rho_A \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{a}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix}
$$

and

$$
\begin{bmatrix} \widehat{y}_t \\ \widehat{i}_t \\ \widehat{l}_t \\ \widehat{c}_t \end{bmatrix} = \begin{bmatrix} S_5 & S_6 \\ S_1 & S_2 \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{a}_t \end{bmatrix}
$$

that is, the solution implies that the vector of log-deviation of control variables is a function of the vector of the state variables, and where the matrices S_5 and S_6 are function on the parameters of the model (α, β, γ, δ, ρ_A, σ_A).   Therefore, the resolution of the model involves the calibration or estimation of the above matrices, i.e., the structural parameters of the model, linking the dynamic of the control variables with the state variables, where the state variables follow an autoregressive vector of order 1. Given the process for the state variables, we can predict its future value, so using the latter system, we can obtain projections for the future value of control variables.

Given the calibrated parameter values, the specific solution for our model would be:

$$
A = \begin{bmatrix} 1.0000 & 0.0000 & -0.6500 \\ 0.0882 & -0.0204 & 0.0000 \\ 1.0000 & 0.0000 & -1.5635 \end{bmatrix}
$$

$$
B = \begin{bmatrix} 0.35 & 0 \\ 0 & 0.0678 \\ 0 & 1 \end{bmatrix}
$$

$$
C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
D = \begin{bmatrix} 0.0882 & 1 \\ 1 & 0 \end{bmatrix}
$$

$$
F = \begin{bmatrix} -0.0882 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$G = \begin{bmatrix} 0 & 1 \\ 0.94 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.06 & 0 \end{bmatrix}$$

Given the above matrices, we can proceed to define the following two matrices J and M:

$$J = \begin{bmatrix} 1.0956 & -0.3847 \\ -0.0365 & 0.9537 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.4447 \\ 0.1201 \end{bmatrix}$$

Applying the Jordan decomposition to matrix J, we obtain:

$$O = \begin{bmatrix} 0.1320 & 0.7570 \\ -0.1320 & 0.2430 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.8866 & 0 \\ 0 & 1.1627 \end{bmatrix}$$

being

$$Q = \begin{bmatrix} 0.1497 \\ -0.0295 \end{bmatrix}$$

Finally, we can compute:

$$S_1 = 0.5433$$

$$S_2 = 0.5709$$

$$S_3 = 0.8866$$

$$S_4 = 0.2251$$

$$S_5 = \begin{bmatrix} 0.2124 \\ -0.8893 \\ -0.2116 \end{bmatrix}$$

$$S_6 = \begin{bmatrix} 1.3054 \\ 3.7513 \\ 0.4698 \end{bmatrix}$$

Therefore, the solution of the model is given by the following two systems of equations:

$$\begin{bmatrix} \widehat{k}_{t+1} \\ \widehat{a}_{t+1} \end{bmatrix} = \begin{bmatrix} 0.8866 & 0.2251 \\ 0 & 0.9500 \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{a}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix}$$

and

$$\begin{bmatrix} \widehat{y}_t \\ \widehat{i}_t \\ \widehat{l}_t \\ \widehat{c}_t \end{bmatrix} = \begin{bmatrix} 0.2124 & 1.3054 \\ -0.8893 & 3.7513 \\ -0.2116 & 0.4698 \\ 0.5433 & 0.5709 \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{a}_t \end{bmatrix}$$